

Published in: *International Mathematics Research Notices* **2006**
(2006), Article ID 61570, 37 pages

ON PERMANENTAL POLYNOMIALS OF CERTAIN RANDOM MATRICES

YAN V FYODOROV

ABSTRACT. The paper addresses the calculation of correlation functions of permanental polynomials of matrices with random entries. By exploiting a convenient contour integral representation of the matrix permanent some explicit results are provided for several random matrix ensembles. When compared with the corresponding formulae for characteristic polynomials, our results show both striking similarities and interesting differences. Based on these findings, we conjecture the asymptotic forms of the density of permanental roots in the complex plane for Gaussian ensembles as well as for the Circular Unitary Ensemble of large matrix dimension.

1. INTRODUCTION

The permanent of an $N \times N$ matrix \mathbf{H} with entries H_{ij} ($i, j = 1, \dots, N$) is defined by

$$(1.1) \quad \text{Per}(\mathbf{H}) = \sum_{\sigma} \prod_{i=1}^N H_{i\sigma(i)}$$

where the sum is taken over all permutations σ of $\{1, \dots, N\}$. In strong contrast to determinants, computing permanents is $\#$ -complete problem [53], and a considerable effort was spent on developing various approximation methods, see [21] and references therein. Permanents have important combinatorial meaning, in particular the permanents of $(0, 1)$ matrices enumerate matchings in bipartite graphs, see e.g. [20]. Permanents also have applications in physics of interacting Bose particles, see e.g. [54]. A concise introduction into properties of permanents can be found in [38], the standard reference is [42].

The permanental polynomial afforded by \mathbf{H} is defined as the permanent of the characteristic matrix, i.e.

$$(1.2) \quad p(\mu) = \text{Per}(\mu \mathbf{1}_N - \mathbf{H}) = \mu^N - a_1 \mu^{N-1} + a_2 \mu^{N-2} - \dots + (-1)^N a_N$$

The present research is supported by EPSRC grant EP/C515056/1 "Random Matrices and Polynomials: a tool to understand complexity".

Preprint AIM 2006-8, American Institute of Mathematics.

where $\mathbf{1}_N$ stands for the $N \times N$ identity matrix. In particular, it is easy to show that $a_1 = \text{Tr} \mathbf{H}$ and $a_N = \text{Per}(\mathbf{H})$. Although a permanental polynomial is not preserved by similarity, it is preserved by permutational similarity:

$$(1.3) \quad \text{Per}(\mu \mathbf{1}_N - \mathbf{P}^{-1} \mathbf{H} \mathbf{P}) = \text{Per}(\mu \mathbf{1}_N - \mathbf{H})$$

for any $N \times N$ permutation matrices \mathbf{P} . Permanental polynomials attracted some interest in graph theory and its applications to chemistry, see e.g. [13, 39, 43]. In particular, some general statements on location of the roots of permanental polynomials (known as permanental roots) can be found in [39, 43]. Some curious series expansion of moments of permanental polynomials were developed in [17].

The goal of the present paper is to initiate research on permanental polynomials of various classes of random matrices. Although some questions concerning permanents of random matrices were investigated earlier, see the books [37, 47] as well as more recent results in [45, 46], the present author is not aware of any related studies on permanental polynomials. The final goal of the whole project should be finding various statistical characteristics of the permanental roots, the mean density in the complex plane being the simplest, yet nontrivial example. Formulated as such the problem seems to be quite challenging, and no results are currently available.

To get some insight into the problem in the present paper we address simpler, yet informative objects which we call the correlation functions of permanental polynomials. The n -point correlation function depends on n complex parameters μ_1, \dots, μ_n and is defined as the expectation value

$$(1.4) \quad \langle \text{Per}(\mu_1 \mathbf{1}_N - \mathbf{H}) \text{Per}(\mu_2 \mathbf{1}_N - \mathbf{H}) \dots \text{Per}(\mu_n \mathbf{1}_N - \mathbf{H}) \rangle_H,$$

where the angular brackets stand for the averaging over the distribution of the matrix entries of \mathbf{H} . Particular classes of the distributions we are dealing with will be discussed explicitly later on.

To this end it is appropriate to mention that in the last few years it became clear that the general correlation functions of characteristic polynomials $d(\mu) = \det(\mu \mathbf{1}_N - \mathbf{H})$ of random matrices of various kinds are extremely informative objects, with a rich mathematical structure and important applications. One of the sources of interest in such objects originated from attempts by Keating and Snaith [33, 34, 35] to get an analytical insight into statistical properties of the Riemann zeta-function and related functions, whose zeroes are believed to share statistically many features with the eigenvalues of unitary (or Hermitian) random matrices. Brezin and Hikami [11, 12] understood that

the correlation functions of products of characteristic polynomials for invariant ensembles of random matrices can be represented as determinants made of orthogonal polynomials generated by the random matrix measure, see (1.10) below. Fyodorov and Strahov generalized this result to correlation functions involving both products and ratios of characteristic polynomials [28, 50] (an elegant proof is by Baik, Deift and Strahov [5], and important extensions to other symmetry classes are due to Borodin and Strahov, [10]). The latter objects are especially important for applications in physics of quantum chaotic systems, see e.g. Andreev and Simons [3], and more recently in Quantum Chromodynamics, see e.g. Verbaarschot and Splittorff [51], Fyodorov and Akemann [26, 2] and references therein. Characteristic polynomials of non-Hermitian/non-unitary random matrices of several types were studied most recently by Akemann and collaborators [4, 1] and by Fyodorov and Khoruzhenko [27].

All these facts providing an additional strong motivation for the research, a number of far-reaching generalizations and extensions of the above-mentioned results were obtained in recent years by various groups, see [8, 9, 14, 15, 16, 31]. In addition, nice combinatorial interpretations of moments of the characteristic polynomials were recently revealed in [19, 23, 49].

A major part of the recent progress in dealing with the characteristic polynomials was essentially possible due to the fact that those polynomials depended only on the eigenvalues $\lambda_1, \dots, \lambda_N$ of random matrices, not on their eigenvectors. The joint probability density (j.p.d.) $\mathcal{P}(\lambda_1, \dots, \lambda_N)$ of those eigenvalues is well-known for the standard classes of random matrices.

As one of the most important examples, consider the space \mathcal{M} of $N \times N$ Hermitian matrices $\mathbf{H} = (H_{ij}) = \mathbf{H}^*$, with the probability measure on \mathcal{M} being chosen according to

$$(1.5) \quad \mathcal{P}(\mathbf{H}) d\mathbf{H} = c e^{-N \text{Tr} V(\mathbf{H})} \prod_{i=1}^N dH_{ii} \prod_{i < j}^N dH_{ij}^R dH_{ij}^I$$

where $H_{ij} = H_{ij}^R + iH_{ij}^I$ in terms of its real and imaginary parts, and c stands for the corresponding normalization constant. The function $V(\mathbf{H})$ is known as the *potential* and is usually assumed to be a polynomial in \mathbf{H} of even power with real coefficients, the coefficient in front of the highest power being positive. Such potentials define the so-called Unitary Ensembles [18], which is a distinguished subset of distributions invariant with respect to unitary conjugations: $\mathbf{H} \rightarrow \mathbf{U} \mathbf{H} \mathbf{U}^*$, with \mathbf{U} standing for any $N \times N$ unitary matrix from the group $U(N)$. Let us

further use the spectral decomposition $\mathbf{H} = \mathbf{U}\Lambda\mathbf{U}^*$ of the Hermitian matrix \mathbf{H} in terms of the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ of its (real) eigenvalues, and some $\mathbf{U} \in U(N)$. This decomposition induces the corresponding decomposition of the measure (see [18, 25, 41] for more detail):

$$(1.6) \quad \mathcal{P}(\mathbf{H})d\mathbf{H} = C_N e^{-N\text{Tr}V(\Lambda)} \Delta^2(\Lambda) \prod_{i=1}^N d\lambda_i d\mu_N(U)$$

where $d\mu_N(U)$ stands for the corresponding invariant (Haar's) measure on $U(N)$ (normalized to unity), $\Delta(\Lambda) = \prod_{i < j} (\lambda_i - \lambda_j)$ is the so-called Vandermonde factor, and C_N is the corresponding normalization constant. Integrating out the U -variables immediately gives j.p.d. of real eigenvalues in the form

$$(1.7) \quad \mathcal{P}(\lambda_1, \dots, \lambda_N) \prod_{i=1}^N d\lambda_i = C_N \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N dw(\lambda_i)$$

where the measure $dw(\lambda)$ is related to the potential $V(\lambda)$ as $dw(\lambda) = e^{-NV(\lambda)}d\lambda$. It is then a straightforward exercise[18, 25] to show that the expectation value $\langle d(\mu) \rangle_H$ of the characteristic polynomial is simply given by

$$(1.8) \quad \langle \det(\mu \mathbf{1}_N - \mathbf{H}) \rangle_H = \pi_N(\mu)$$

where $\pi_k(x)$ stands for the k -th monic orthogonal polynomial ($\pi_k(x) = x^k + \text{lower powers}$), with the orthogonality being understood with respect to the measure $dw(x)$ on the real axis:

$$(1.9) \quad \int \pi_j(x) \pi_k(x) dw(x) = c_j c_k \delta_{jk}$$

In fact, the relation equivalent to (1.8) was known in classical theory of orthogonal polynomials (although, without any reference to random matrices) for more than one hundred years[48]. It is its modern random matrix interpretation that made evident a deep relation between the random characteristic polynomials and orthogonal polynomials, and enabled those far-reaching generalizations that were already mentioned above. In particular, the simplest case of the Brezin-Hikami result reads [11]:

$$(1.10) \quad \langle d(\mu_1) d(\mu_2) \rangle_{GUE} = \frac{1}{\mu_1 - \mu_2} \det \begin{pmatrix} \pi_N(\mu_1) & \pi_N(\mu_2) \\ \pi_{N+1}(\mu_1) & \pi_{N+1}(\mu_2) \end{pmatrix},$$

Naively one may expect that nothing similar is to be valid for permanental polynomials, precisely because of the lack of invariance under general similarity transformations. This fact makes seemingly impractical presenting permanental polynomials in terms of the matrix eigenvalues, and forces us to search for an alternative technique of evaluating the random matrix averages featuring in (1.4), both over the eigenvalues and the corresponding "angular" variables (=eigenvectors). To this end, we start with revealing a possibility to represent permanents as a multiple contour integral, see Lemma (2.1). Such a representation turned out to be very useful, and allowed us to calculate explicitly the expectation value of the permanental polynomial for a general Unitary Ensemble of Hermitian random matrices. In fact, integrating out the "angular" variables can be done explicitly with the help of a variant of the famous HarishChandra-Itzykson-Zuber (HCIZ) formula [30, 32], and is based essentially on interpreting that formula in terms of the Schur function expansion method, see e.g. [44, 6]. It came as a somewhat surprising result that the expectation value of the permanental polynomial can be generally expressed as a one-fold integral of the same monic orthogonal polynomials featuring in (1.8), see (1.11). In fact, we prove the following

Theorem 1.1. *The expected value of the permanental polynomial for random matrices taken from a Unitary Ensemble characterized by a potential $V(x)$ is given by*

$$(1.11) \quad \begin{aligned} \langle p(\mu) \rangle &= a_N^{-1} \int e^{-NV(\lambda)} (\mu - \lambda)^{2N-1} \pi_{N-1}(\lambda) d\lambda \\ &= a_N^{-1} \sum_{k=0}^N a_k \mu^k, \quad a_k = \frac{(2N-1)!}{N!(N-1)!} \int e^{-NV(\lambda)} \lambda^{2N-1-k} \pi_{N-1}(\lambda) d\lambda \end{aligned}$$

where $\pi_N(\lambda) = \lambda^N + \text{lower degrees}$ is the N -th monic polynomial orthogonal with respect to the measure $dw(x) = e^{-NV(x)} dx$ on the real axis.

For a particular case of the Gaussian Unitary Ensemble (GUE) characterized by the measure $V(x) = x^2/2$ the integral in (1.11) can be evaluated explicitly yielding, even more surprisingly, the relation almost identical to (1.8), with the only simple change $\mu \rightarrow -i\mu$, see (3.20):

$$(1.12) \quad \langle p(\mu) \rangle_{GUE} = i^N \langle d(-i\mu) \rangle_{GUE}, \quad d(\mu) = \det(\mu \mathbf{1}_N - H)$$

Note: In this work we consider general Hermitian matrices with the eigenvalues $\lambda_i \in (-\infty, \infty)$. Sometimes, however, one may wish

to deal with positive definite Hermitian matrices, and thus to confine the eigenvalues to the positive semi-axis $\lambda_i \in [0, \infty)$ (e.g. the so-called Wishart (or Laguerre) Ensemble, see [22], with $V(\lambda) = \lambda$, $\lambda > 0$). Further choices confining eigenvalues to an interval of the real axis are also possible. Extensions of our result to all those cases are self-evident.

The explicit evaluation of the lowest non-trivial correlation function of permanental polynomials turns out to be possible also for one more important class of random matrices, those from the Circular Unitary Ensemble (CUE), see Section 5. The corresponding matrices are general unitary $N \times N$ and are considered to be uniformly distributed on the group manifold according to the corresponding Haar's measure. The calculation is based on a recent progress in evaluating some integrals over $U(N)$ [27, 52], and again has the Schur function expansion in its heart.

Although in principle the Schur function expansion method can be extended to obtain higher-order correlation functions, the actual calculations for general invariant ensembles as well as for the Circular Unitary Ensemble become cumbersome and the progress is yet to be achieved. We therefore concentrate on two particular cases where a progress can be achieved by a different method: the Gaussian Unitary Ensemble of Hermitian matrices (Section 4) and the so-called Ginibre Ensemble (Section 6) of general complex matrices with independent, identically distributed normal entries. Our starting expression is the same contour integral representation of a permanent, but the specific choice of the measure allows to perform the ensemble averaging in a closed form without resorting to HCIZ formula. Further progress is based on a method frequently employed in the physics of disordered systems and known there as the Hubbard-Stratonovich transformation, see e.g.[24]. In this way we managed to represent the general n -point correlation function via n -fold integrals:

Theorem 1.2. *For any integer $n = 1, 2, \dots$ the general n -point correlation function of permanental polynomials of GUE matrices has the following integral representation:*

$$(1.13) \quad \left\langle \prod_{k=1}^n p(\mu_k) \right\rangle_{GUE} = \int e^{-\frac{N}{2} \text{Tr} \mathbf{q}^2} [\text{Per}(\mathbf{M} - \mathbf{q})]^N d\mathbf{q}$$

In this expression $\mathbf{M} = \text{diag}(\mu_1, \dots, \mu_n)$ is $n \times n$ diagonal matrix and \mathbf{q} is a general $n \times n$ Hermitian matrix.

Note. For $\mu_1 = \mu_2 = \dots = \mu_N \equiv \mu$ the identity (1.13) can be written as

$$(1.14) \quad \int e^{-\frac{N}{2} \text{Tr} \mathbf{H}^2} [\text{Per}(\mu \mathbf{1}_N - \mathbf{H})]^n d\mathbf{H} = \int e^{-\frac{N}{2} \text{Tr} \mathbf{q}^2} [\text{Per}(\mu \mathbf{1}_n - \mathbf{q})]^N d\mathbf{q}$$

Similar identities are known to hold for moments of GUE characteristic polynomials, and are frequently called "the duality relations", see e.g. [28, 22] and references therein. In that context they were commonly thought to be intimately connected to the underlying integrable structures (Toda lattice hierarchies, see e.g. [22]). The fact that they emerge in the present context may indicate some hidden integrability lurking behind the problem of permanental polynomials.

The formula (1.13) is already a substantial simplification, especially if we have in mind random matrices of large size N , so that n can be small in comparison with the size of the matrix. Nevertheless, to get a closed form result for any n and N amounts to evaluating those integrals and remains an outstanding problem. However, for the two-point correlation functions $n = 2$ the integral can be evaluated explicitly, and the result is expressed in a very attractive "determinantal" form which is actually again almost identical to (1.10) for characteristic polynomials. Namely, we found that

$$(1.15) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \langle d(-i\mu_1) d(i\mu_2) \rangle_{GUE}.$$

We also show that a formula similar to (1.13) can be derived by essentially the same method for the Ginibre Ensemble, see (6.3). Finally, the same method works also for the ensemble of real symmetric matrices with the Gaussian probability measure (the so-called Gaussian Orthogonal Ensemble, GOE), for which we again obtain explicit expressions for the one- and two-point correlation functions of permanental polynomials, see Section 7. Close similarities with the corresponding results obtained earlier [12] for characteristic polynomials of GOE matrices are again apparent. In particular, both the relations (1.12) and (1.15) remain valid.

The last (but by far not the least) point to be mentioned is that one is usually interested in the asymptotic, large- N behaviour of random matrix characteristics. In particular, for GUE characteristic polynomials asymptotic behaviour is very different for $\mu \in (-2, 2)$ (the so-called "bulk scaling" regime), for $|\mu| = 2 + O(N^{-2/3})$ ("soft edge scaling") and for all other complex values of μ [18]. This is naturally related to the fact that all N eigenvalues λ_i of GUE matrices, being real, concentrate in the limit $N \rightarrow \infty$ in the interval $\lambda \in [-2, 2]$ with probability tending to one. The corresponding limiting density of eigenvalues is given

by the famous Wigner semicircular law: $\rho_\infty(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$. Moreover, within that interval those eigenvalues are non-trivially correlated on the distances $|\lambda_i - \lambda_j|$ of the order of $O(1/N)$. The latter fact is reflected, in particular, in the existence of the following limit, [11], see also [25]:

$$(1.16) \quad \lim_{N \rightarrow \infty} \frac{\langle d(\mu_1) d(\mu_2) \rangle_{GUE}}{\langle d^2(\mu_1) \rangle_{GUE}} = K_\infty[N\rho_\infty(\mu)(\mu_1 - \mu_2)], \quad K_\infty[x] = \frac{\sin x}{x}$$

whenever $\mu_1, \mu_2 \in (-2, 2)$, $\lim_{N \rightarrow \infty} N(\mu_1 - \mu_2) < \infty$. Moreover, the limiting expression (frequently called the "Dyson kernel") is not restricted by GUE, but is valid for a very broad class of Unitary Invariant Ensembles of random Hermitian matrices ("universality").

Adopting those facts to the case of GUE permanental polynomials by the relation (1.12) we see a special role of the interval $\mu \in [-2i, 2i]$ of the imaginary axis. Combining (1.15) and (1.16) we also see that the values of permanental polynomials close to the points $i\mu$ and $-i\mu$ are non-trivially correlated in the limit of large N , as long as $\mu \in (2, 2)$. The latter fact could probably be related to the property of the permanental roots to come in complex conjugate pairs (as is evident from coefficients of the permanental polynomials of Hermitian matrices being real). All this facts taken together make it rather plausible to expect permanental roots of large GUE matrices to accumulate asymptotically in the vicinity of the interval $[-2i, 2i]$ of the imaginary axis. Below we provide an argument in favour of the validity of the following

Conjecture: *The normalized asymptotic limiting density of the permanental roots of large GUE matrices in the complex plane $z = x + iy$ is vanishing outside the interval $[-2i, 2i]$ of the imaginary axis. Inside that interval it is given by the Wigner semicircular law.*

Our argument goes as follows. For $z = x + iy$ define the function

$$(1.17) \quad \Phi(x, y) = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \ln |p(z)|^2 \rangle_{GUE}$$

According to the standard potential theory [36] the expectation value of the limiting density $\rho(x, y)$ of permanental roots in the complex plane is given by $\langle \rho(x, y) \rangle_{GUE} = \frac{1}{4\pi} \Delta \Phi(x, y)$, with Δ standing for the Laplacian in variables x and y (understood in the sense of distributions). Unfortunately, performing the ensemble average in (1.17) is an outstanding problem. Assume however that the operations of taking the ensemble average and taking the logarithm commute in the limit

$N \rightarrow \infty$, i.e.

$$(1.18) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \langle \ln |p(z)|^2 \rangle_{GUE} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |p(z)|^2 \rangle_{GUE}$$

Such commutativity is indeed known to occur, for example, in the case of characteristic polynomials of GUE matrices[7], and one may hope it to hold for permanental polynomials as well. It would also imply the self-averaging property: the asymptotic root density is given by its expectation value. Proving relation (1.18) for GUE permanental polynomials remains an outstanding problem. If however such a relation holds, the function $\Phi(x, y)$ can be found from (1.18), (1.15) and (1.10) by exploiting the known Plancherel-Rotach asymptotics for the Hermite polynomials $\pi_N(\mu)$, see e.g. Eq.(7.93) of [18]. A simple calculation yields for $x \neq 0$

$$(1.19) \quad \Phi(x, y) = \Psi(y - ix) + \Psi(y + ix)$$

$$(1.20) \quad \Psi(q) = \frac{1}{8}(q - \sqrt{q^2 - 4})^2 - \ln(q - \sqrt{q^2 - 4}) + const$$

For $x = Im q \neq 0$ the function $\Psi(q)$ is obviously analytic in the plane of complex variable $q = y + ix$, hence the function $\Phi(x, y)$ is harmonic resulting in vanishing root density $\rho(x, y) = 0$. Thus, the only non-vanishing root density is possible for $x = 0$, i.e. along the imaginary axis in the original complex plane $z = x + iy$. To calculate this density from $\Phi(x, y)$ given in (1.19) one again follows the standard potential theory and evaluates the jump of the normal derivative: $\frac{\partial}{\partial x} \Phi(x, y)|_{x \rightarrow 0^+} - \frac{\partial}{\partial x} \Phi(x, y)|_{x \rightarrow 0^-}$ across the line $x = 0$. It immediately yields the Wigner semicircular law in the interval $[-2i, 2i]$ of the imaginary axis, in agreement with the proposed conjecture. Similar, but slightly more technically involved calculations suggest validity of the semicircular density profile at the interval $[-2\sqrt{2}i, 2\sqrt{2}i]$ of the imaginary axis for the permanental roots of Gaussian Orthogonal Ensemble of real symmetric random matrices.

The relation (1.18) is known to hold also for characteristic polynomials of Ginibre Ensemble and Circular Unitary Ensemble (CUE) matrices, see [27] and the discussion at the end of Sections 5 and 6. Assuming its validity for permanental polynomials of those ensembles one can conjecture the ensuing asymptotic densities of the permanental roots for those ensembles inside the unit circle $x^2 + y^2 \leq 1$ in the

complex plane. They are given by:

$$(1.21) \quad \rho(x, y) = \frac{2}{\pi} \frac{1}{(1 + x^2 + y^2)^2}, \quad \text{CUE}$$

$$(1.22) \quad \rho(x, y) = \frac{1}{\pi}, \quad \text{Ginibre Ensemble}$$

and vanishing density outside the unit circle. Whereas for Ginibre ensemble the conjectured density of permanental roots (1.22) is the same as that for the characteristic roots, the density profile (1.21) is obviously radically different from the CUE eigenvalue density concentrated on the unit circle. Apart from investigating the validity of the above conjectures, any result on correlation patterns typical for permanental roots of large random matrices of various types are presently lacking. Finally, the question whether an accumulation of permanental roots in the vicinity of the imaginary axis is generic for more general ensembles of Hermitian random matrices is very intriguing and clearly deserve to be a subject of a separate study.

Acknowledgements. The author is grateful to Boris Khoruzhenko for numerous stimulating discussions at various stages of this work, and in particular for providing access to the references [37] and [7]. Alexander Gamburd is acknowledged for discussion of results at early stages of the research, and Ilya Krasikov for a few useful remarks and encouragement. Although first results in this direction were obtained in 2004, the paper would be hardly completed, had it been not for the author's participation in January 2006 at the workshop "Random Analytic Functions" at American Institute of Mathematics, Palo Alto. The author is very grateful to the organizers, coordinators and participants of the workshop for numerous informative discussions and inspiring atmosphere, in particular to Pavel Blecher, Brian Conrey, David Farmer and especially to Stanislav Molchanov for his remarks on the conjecture concerning the asymptotic density of permanental roots.

2. PERMANENTS AS CONTOUR INTEGRALS

Let $\mathbf{z} = (z_1, \dots, z_N)^T$ stand for N -component complex (column) vector, and similarly define a (row) vector $\xi^* = (\bar{\xi}_1, \dots, \bar{\xi}_N)$, with the bar standing for the complex conjugation, and the star for Hermitian conjugation. Further denote $\mathbf{z} \otimes \xi^*$ the $N \times N$ matrix with entries $z_i \bar{\xi}_j$, $i, j = 1, \dots, N$, and let $(\xi^* \mathbf{z})$ stand for the scalar product $\sum_{i=1}^N \bar{\xi}_i z_i$, and $\xi^* \mathbf{F} \mathbf{z}$ for the bilinear form $\sum_{ij} F_{ij} \bar{\xi}_i z_j$.

Lemma 2.1. *The permanent of an arbitrary $N \times N$ matrix $\mathbf{F} = \{F_{ij}\}$ can be expressed through the contour integral as*

$$(2.1) \quad \text{Per } \mathbf{F} = \frac{1}{(2\pi)^{2N}} \oint_{|z_1|=1} \cdots \oint_{|z_N|=1} \oint_{|\xi_1|=1} \cdots \oint_{|\xi_N|=1} \\ \times \exp \text{Tr} [\mathbf{F} \mathbf{z} \otimes \xi^*] \prod_{k=1}^N \frac{dz_k}{z_k^2} \frac{d\bar{\xi}_k}{\bar{\xi}_k^2}$$

Proof. Obviously, $\text{Tr} [\mathbf{F} \mathbf{z} \otimes \xi^*] = \xi^* \mathbf{F} \mathbf{z}$ for any matrix \mathbf{F} . Denote $\mathbf{q} = \mathbf{F} \mathbf{z}$, so that $\exp (\xi^* \mathbf{F} \mathbf{z}) = \prod_{i=1}^N \exp \bar{\xi}_i q_i$. Use $\frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\bar{\xi}}{\bar{\xi}^2} \exp \bar{\xi} q = -q$ to see that the right-hand side of (2.1) is given by

$$(2.2) \quad (-i)^N \frac{1}{(2\pi)^N} \oint_{|z_1|=1} \cdots \oint_{|z_N|=1} \prod_{i=1}^N \sum_{j=1}^N F_{ij} z_j \prod_{k=1}^N \frac{dz_k}{z_k^2} \\ = \frac{1}{(2\pi i)^N} \sum_{j_1=1, \dots, j_N=1}^N F_{1j_1} \cdots F_{Nj_N} \oint_{|z_1|=1} \cdots \oint_{|z_N|=1} z_{j_1} z_{j_2} \cdots z_{j_N} \prod_{k=1}^N \frac{dz_k}{z_k^2}$$

The last integral is obviously non-vanishing only as long as $j_1 \neq j_2 \neq \dots \neq j_N$, and the required relation immediately follows. \square

Our way of representing permanents as contour integrals is of central importance for the rest of the paper. Despite its apparent simplicity, we were not able to trace such a formula in the available literature on permanents. In the Appendix we demonstrate how the formula (2.1) generates a known integral representation of the permanent of a positive definite matrix as a multivariate Gaussian integral used earlier in the physical applications [54].

In what follows we will systematically use the short-hand notation

$$(2.3) \quad \oint [\dots] \mathcal{D}_N(\mathbf{z}, \xi^*) \\ \equiv \frac{1}{(2\pi)^{2N}} \oint_{|z_1|=1} \cdots \oint_{|z_N|=1} \oint_{|\xi_1|=1} \cdots \oint_{|\xi_N|=1} [\dots] \prod_{k=1}^N \frac{dz_k}{z_k^2} \prod_{k=1}^N \frac{d\bar{\xi}_k}{\bar{\xi}_k^2}$$

In particular, we will make use of the following

Lemma 2.2. *Let $f(z)$ be an entire function of the complex variable z represented by its convergent power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Then:*

$$(2.4) \quad I_N(f) = \oint f(\xi^* \mathbf{z}) \mathcal{D}_N(\mathbf{z}, \xi^*) = f_N$$

Proof. Use $z_k = e^{i\theta_k}$, $\xi_k = e^{i\psi_k}$, $k = 1, \dots, N$, where $0 \leq \theta_k, \psi_k < 2\pi$, so that

$$I_N(f) = \left(\frac{i}{2\pi}\right)^{2N} \int_0^{2\pi} \int_0^{2\pi} e^{-i \sum_k (\theta_k - \psi_k)} f\left(\sum_k e^{i(\theta_k - \psi_k)}\right) \prod_k d\theta_k \prod_k d\psi_k$$

Shifting $\theta_k \rightarrow \alpha_k = \theta_k - \psi_k$, integrating out ψ_k and further reinstating the unimodular complex variable $z = e^{i\alpha_k}$ we see that

$$I_N(f) = \left(\frac{i}{2\pi}\right)^N \oint_{|z_1|=1} \dots \oint_{|z_N|=1} f\left(\sum_k z_k\right) \frac{dz_1}{z_1^2} \dots \frac{dz_N}{z_N^2}$$

The integrations can be easily performed using the power series expansion, and the result follows. \square

3. MEAN VALUE OF THE PERMANENTAL POLYNOMIALS FOR UNITARY INVARIANT ENSEMBLES OF HERMITIAN MATRICES

The main goal of the present section is to verify the formula (1.11). We need the following

Lemma 3.1. *Let Γ be any $N \times N$ matrix such that it has precisely one non-zero eigenvalue γ , the rest $N - 1$ eigenvalues being equal to zero. Then the following identity holds:*

$$(3.1) \quad I(\beta; \Lambda, \Gamma) = \int_{U(N)} \exp \beta \text{Tr} [\mathbf{U} \Lambda \mathbf{U}^* \Gamma] d\mu_N(U)$$

$$(3.2) \quad = \sum_{n=0}^{\infty} \frac{(N-1)!}{(N+n-1)!} (\beta\gamma)^n h_n(\lambda_1, \dots, \lambda_N)$$

$$(3.3) \quad = \frac{(N-1)!}{(\beta\gamma)^{N-1}} \sum_{i=1}^N \frac{e^{\beta\gamma\lambda_i}}{\prod_{j \neq i} (\lambda_j - \lambda_i)}$$

where

$$(3.4) \quad h_n(\lambda_1, \dots, \lambda_N) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq N} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n}$$

are the complete symmetric functions.

Remark: The integral in the right hand side of (3.1) is the well-known HarishChandra-Itzykson-Zuber (HCIZ) integral [30, 32]. In the case of a general Hermitian matrix Γ with all distinct non-zero real eigenvalues $\gamma_1, \dots, \gamma_N$ the result was found by Itzykson and Zuber [32] in the form

$$(3.5) \quad I(\beta; \Lambda, \Gamma) = \beta^{-N(N-1)/2} \left(\prod_{n=0}^{N-1} n! \right) \frac{\det e^{\beta\lambda_i\gamma_j}}{\Delta(\Lambda)\Delta(\Gamma)}.$$

In our case it is most convenient to exploit that fact that one can recover HCIZ expression by the Schur function expansion method[44] (also known as the character expansion, see [6]). In particular, the method does not require matrices Λ and Γ to be Hermitian, and we will indeed consider them to be general complex.

Proof. Let us shortly remind the definition and basic properties of the Schur functions (see the book [40] for more detail). A partition $r = (n_1, n_2, \dots, n_N)$ is a finite sequence of integers, called "parts", satisfying $(n_1 \geq n_2 \geq \dots \geq n_N)$, which is characterized by its weight $|r| = \sum_j n_j$ and its length $l(r)$ equal to the number of non-zero parts. With any partition of the length $l(r) \leq N$ one associates the Schur function

$$(3.6) \quad s_r(x_1, x_2, \dots, x_N) = \frac{\det \left(x_i^{n_j + N - j} \right)_1^N}{\det \left(x_i^{N - j} \right)_1^N}$$

which is a symmetric polynomial in x_1, x_2, \dots, x_N homogeneous of degree $|r|$. By convention, $s_r(x_1, \dots, x_N) = 0$ for $l(r) > N$.

For the simplest case of partitions of length one $r = (n)$ (i.e. $n_1 = n, n_2 = \dots = n_N = 0$) the Schur functions coincide with the complete symmetric functions:

$$(3.7) \quad s_{(n)}(x_1, \dots, x_N) = h_n(x_1, \dots, x_N)$$

defined in (3.4). Using definition (3.6), expanding the determinant in the numerator in entries of the first column and remembering

$$\det \left(x_i^{N - j} \right)_1^N = \prod_{i < j} (x_i - x_j)$$

we easily get a useful identity:

$$(3.8) \quad (-1)^{N-1} \sum_{i=1}^N \frac{x_i^{n+N-1}}{\prod_{j \neq i} (x_j - x_i)} = \begin{cases} 0, & n = -(N-1), -(N-2), \dots, -1 \\ h_n(x_1, \dots, x_N), & n = 0, 1, 2, \dots \end{cases}$$

For any $N \times N$ matrix $\mathbf{M} \in Gl(N)$ one defines Schur functions of the matrix argument, $s_r(\mathbf{M})$, as

$$(3.9) \quad s_r(\mathbf{M}) = s_r(m_1, \dots, m_N)$$

where m_k are eigenvalues of the matrix \mathbf{M} . The Schur functions of matrix argument are the characters of irreducible representations of the general linear group (and its unitary subgroup), and as a consequence have the important property of orthogonality. If r_1 and r_2 are two

partitions, and $\mathbf{M}_1, \mathbf{M}_2$ are two $N \times N$ matrices, then (see p.445 of [40])

$$(3.10) \quad \int_{U(N)} s_{r_1}(\mathbf{U}\mathbf{M}_1) \overline{s_{r_2}(\mathbf{U}\mathbf{M}_2)} d\mu_N(U) = \delta_{r_1, r_2} \frac{s_{r_1}(\mathbf{M}_1 \mathbf{M}_2^*)}{s_{r_1}(\mathbf{1}_N)}$$

and

$$(3.11) \quad \int_{U(N)} s_r(\mathbf{U}\mathbf{M}_1 \mathbf{U}^* \mathbf{M}_2) d\mu_N(U) = \frac{s_r(\mathbf{M}_1) s_r(\mathbf{M}_2)}{s_r(\mathbf{1}_N)}$$

Using all these facts, one can calculate the integral in (3.1) by expanding the integrand in a series with respect to Schur functions, and performing the integration over the unitary group using (3.11). After working out the coefficients of expansion the result is expressed as a sum over partitions [6, 44]

$$(3.12) \quad I(\beta; \Lambda, \Gamma) = \sum_r \beta^{n_1 + \dots + n_N} \prod_{i=1}^N \frac{(N-i)!}{(N+n_i-i)!} s_r(\Lambda) s_r(\Gamma)$$

for general matrices Λ and Γ . In our particular case $\gamma_2 = \dots = \gamma_N = 0$, and therefore $s_r(\Gamma) = s_r(\gamma_1) = \gamma_1^n$ for $r = (n)$, i.e for partitions of the length $l(r) = 1$, and $s_r(\Gamma) = 0$ for $l(r) > 1$. Using (3.7) the expression (3.12) immediately takes the form of (3.2), and the equivalence to (3.3) follows after exploiting the identity (3.8). \square

Now we are in position to prove **Theorem (1.1)**.

Proof. Our starting point is the integral representation (2.1) according to which the permanent polynomial $p(\mu) = \text{Per}(\mu \mathbf{1}_N - \mathbf{H})$ is given by

$$(3.13) \quad p(\mu) = \oint \exp \mu(\xi^* \mathbf{z}) \exp -\text{Tr} [\mathbf{H} \mathbf{z} \otimes \xi^*] \mathcal{D}_N(\mathbf{z}, \xi^*)$$

According to (1.6), (1.7) the ensemble average of this polynomial then amounts to evaluating the integral

$$(3.14) \quad \langle p(\mu) \rangle = C_N \int_{-\infty}^{\infty} e^{-N \text{Tr} V(\Lambda)} \Delta^2(\Lambda) \prod_{i=1}^N d\lambda_i \oint e^{\mu(\xi^* \mathbf{z})} J_N(\mathbf{z}, \xi^*) \mathcal{D}_N(\mathbf{z}, \xi^*)$$

where

$$(3.15) \quad J_N(\mathbf{z}, \xi^*) = \int_{U(N)} \exp -\text{Tr} [\mathbf{U} \Lambda \mathbf{U}^* \mathbf{z} \otimes \xi^*] d\mu_N(U)$$

For performing the latter integration over the unitary group we can use the Lemma (3.1) due to the fact that the (non-Hermitian) matrix $\mathbf{z} \otimes \xi^*$ has precisely one non-trivial eigenvalue equal to the scalar product $(\xi^* \mathbf{z})$. We find

$$(3.16) \quad J_N(\mathbf{z}, \xi^*) = \sum_{n=0}^{\infty} \frac{(N-1)!}{(N+n-1)!} (-1)^n (\xi^* \mathbf{z})^n h_n(\lambda_1, \dots, \lambda_N)$$

Substituting this back to (3.14) we at the next step should evaluate the contour integral according to (2.4). For doing this we need to find N -th term in the Taylor expansion of $e^{\mu(\xi^* \mathbf{z})} J_N(\mathbf{z}, \xi)$ in powers of $(\xi^* \mathbf{z})$. This is obviously given by

$$(3.17) \quad \sum_{l=0}^N \frac{\mu^{N-l}}{(N-l)!} \frac{(N-1)!}{(N+l-1)!} (-1)^l h_l(\lambda_1, \dots, \lambda_N)$$

$$= \frac{(N-1)!}{(2N-1)!} \sum_{i=1}^N \frac{1}{\prod_{j \neq i} (\lambda_j - \lambda_1)} \sum_{l=-(N-1)}^N \frac{(2N-1)!}{(N-l)!(N+l-1)!} (-1)^{l+N-1} \mu^{N-l} \lambda_i^{l+N-1}$$

where we exploited the identity (3.8), which allowed us to replace the lower limit $l = 0$ in the second sum in (3.17) with $l = -(N-1)$. Now the sum over l gives simply $(\mu - \lambda_i)^{2N-1}$, and we arrive at the result of the contour integration in the form

$$(3.18) \quad \oint e^{\mu(\xi^* \mathbf{z})} J_N(\mathbf{z}, \xi) \mathcal{D}_N(\mathbf{z}, \xi^*) = \frac{(N-1)!}{(2N-1)!} \sum_{i=1}^N \frac{(\mu - \lambda_i)^{2N-1}}{\prod_{j \neq i} (\lambda_j - \lambda_i)}$$

Next step is to substitute (3.18) to (3.14), and to use the symmetry of the integrand with respect to a permutation of variables $\lambda_1, \dots, \lambda_N$. This gives

$$(3.19) \quad \langle p(\mu) \rangle = \tilde{C}_N \int e^{-N \sum_{k=1}^N V(\lambda_k)} \frac{(\mu - \lambda_1)^{2N-1}}{\prod_{j \neq 1} (\lambda_j - \lambda_1)} \prod_{j \neq i} (\lambda_j - \lambda_i)^2 \prod_{i=1}^N d\lambda_i$$

$$= \tilde{C}_N \int e^{-NV(\lambda_1)} (\mu - \lambda_1)^{2N-1} d\lambda_1$$

$$\times \int e^{-N \sum_{k=2}^N V(\lambda_k)} \prod_{j=2}^N (\lambda_1 - \lambda_j) \prod_{2 \leq j \neq i \leq N} (\lambda_j - \lambda_i)^2 \prod_{i=2}^N d\lambda_i$$

where the particular value of the proportionality factor \tilde{C}_N is not needed for our purposes, and can be established whenever necessary (see below).

According to the classical theory of orthogonal polynomials [48, 18, 22], the integral in the last line of (3.19) is proportional (see (1.8)) to the monic orthogonal polynomial: $\pi_{N-1}(\lambda_1) = \lambda_1^{N-1} + \text{lower degrees}$, defined in (1.9). The overall proportionality constant can be fixed by the obvious condition $\langle p(\mu) \rangle = \mu^N + \text{lower order terms}$, and we immediately arrive at the statement of the Theorem (1.1). \square

Corollary 3.2. *For the Gaussian Unitary Ensemble (GUE) corresponding to the choice $V(x) = x^2/2$ the relation (1.11) takes the form*

$$(3.20) \quad \langle p(\mu) \rangle_{GUE} = i^N \pi_N(-i\mu),$$

Proof. The orthogonal polynomials are obviously Hermite polynomials. The corresponding monic polynomials $\pi_k(x)$ have the following convenient integral representation, see e.g p.53 of [25]:

$$(3.21) \quad \pi_k(x) = (-i)^k \sqrt{\frac{N}{2\pi}} e^{Nx^2/2} \int_{-\infty}^{\infty} q^k e^{-N\frac{q^2}{2} + iNxq} dq$$

Substitute (3.21) to (1.11) and change $\mu - \lambda \rightarrow x$, obtaining

$$\begin{aligned} \langle p(\mu) \rangle_{GUE} &= -\frac{i^{N-1}}{a_N} \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} x^{2N-1} dx \int_{-\infty}^{\infty} q^{N-1} e^{-N\frac{q^2}{2} + iN(\mu-x)q} dq \\ &= \frac{(-1)^N}{a_N N^{N-1}} \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} x^{2N-1} \frac{d^{N-1}}{dx^{N-1}} \left(\int_{-\infty}^{\infty} e^{-N\frac{q^2}{2} + iN(\mu-x)q} dq \right) dx \\ &= \frac{(-1)^N}{a_N N^{N-1}} \int_{-\infty}^{\infty} x^{2N-1} \frac{d^{N-1}}{dx^{N-1}} \left(e^{-\frac{N}{2}(\mu-x)^2} \right) dx \\ (3.22) \quad &= -\frac{1}{a_N} \frac{(2N-1)!}{N^N (N-1)!} e^{-\frac{N}{2}\mu^2} \int_{-\infty}^{\infty} x^N e^{-\frac{N}{2}x^2 + N\mu x} dx \end{aligned}$$

The same method can be used to find the normalization constant from (1.11) and (3.21):

$$(3.23) \quad a_N = \frac{1}{N^N} \frac{(2N-1)!}{(N-1)!} \sqrt{\frac{2\pi}{N}}$$

Comparing (3.22, 3.23) with (3.21) immediately verifies (3.20). \square

In the next section we will be able to reproduce the above result for GUE case by a different method, which is essentially originating from applications of random Gaussian matrices in physics of disordered systems, see [24] and references therein. In this way we will be able also to get some insight into higher order correlation functions of GUE permanental polynomials.

4. CORRELATION FUNCTIONS FOR PERMANENTAL POLYNOMIALS OF GUE MATRICES

The joint probability density of matrix entries for GUE $N \times N$ Hermitian matrices \mathbf{H} is given by

$$(4.1) \quad \mathcal{P}_{GUE}(\mathbf{H})d\mathbf{H} = C_{GUE} \prod_{i=1}^N e^{-\frac{N}{2}H_{ii}^2} \prod_{i < j}^N e^{-N\overline{H}_{ij}H_{ij}} \prod_{i=1}^N dH_{ii} \prod_{i < j}^N dH_{ij}^R dH_{ij}^I$$

where $C_{GUE} = 2^{-\frac{N}{2}} (N/\pi)^{N^2/2}$ is the corresponding normalization constant.

Lemma 4.1. *The identity*

$$(4.2) \quad \langle e^{-Tr\mathbf{H}\mathbf{A}} \rangle_{GUE} = e^{\frac{1}{2N}Tr\mathbf{A}^2}$$

holds for a general $N \times N$ matrix \mathbf{A} with complex entries.

Proof. The integral in the left hand side factorizes into a product of simple Gaussian integrals of the sorts

$$\int e^{-\frac{N}{2}H_{ii}^2 - H_{ii}A_{ii}} dH_{ii} = \sqrt{\frac{2\pi}{N}} \exp \left[\frac{1}{2N}A_{ii}^2 \right]$$

and

$$\int e^{-N\overline{H}_{ij}H_{ij} - H_{ij}A_{ji} - \overline{H}_{ij}A_{ij}} dH_{ij}^{(R)} dH_{ij}^{(I)} = \frac{\pi}{N} \exp \left[\frac{1}{N}A_{ij}A_{ji} \right]$$

The product of all those factors together with the normalization constant C_{GUE} yields the right-hand side of (4.2). \square

The use of this lemma allows one to represent the ensemble averaging of the GUE permanental polynomial $p(\mu)$ from (3.13) in the form

$$(4.3) \quad \langle p(\mu) \rangle_{GUE} = \oint \exp \left[\mu(\xi^*\mathbf{z}) + \frac{1}{2N}(\xi^*\mathbf{z})^2 \right] \mathcal{D}(\mathbf{z}, \bar{\xi})$$

where we used that $Tr(\mathbf{z} \otimes \xi^*)^2 = (\xi^*\mathbf{z})^2$.

Let us now note that the identity (4.2) for the particular case of matrices \mathbf{H} of the size 1×1 can be written in the form:

$$(4.4) \quad \exp \left[\frac{1}{2N}a^2 \right] = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{N}{2}q^2 - qa} dq$$

valid for any complex a and $N > 0$ (we used an obvious replacement $H_{ii} \rightarrow q$, $A_{ii} \rightarrow a$ and a simple rescaling of the integration variable with parameter N). We are going to use this formula for $a = (\xi^*\mathbf{z})$, in order to replace the factor $\exp \left[\frac{1}{2N}(\xi^*\mathbf{z})^2 \right]$ in the integrand of (4.3) with the corresponding Gaussian integral. Such a procedure (and its

generalizations, see below (4.12)) is customarily referred to as "the Hubbard-Stratonovich transformation" in physics[24].

Changing the order of integration, and exploiting (2.1) for the case $\mathbf{F} = (\mu - q)\mathbf{1}_N$, i.e.

$$\oint \exp [(\mu - q)(\xi^* \mathbf{z})] \mathcal{D}(\mathbf{z}, \xi^*) = \text{Per}[(\mu - q)\mathbf{1}_N] = (\mu - q)^N$$

we find

$$(4.5) \quad \langle p(\mu) \rangle_{GUE} = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{N}{2}q^2} (\mu - q)^N dq$$

which in view of (3.21) is indeed equivalent to (3.20).

A nice feature of the present method is that it provides, after a simple generalization, an access to higher-order correlation functions of permanent polynomials for GUE matrices. In fact, we will be able to reduce for any integer $n = 1, 2, \dots$ the general n - point correlation function of permanent polynomials of GUE matrices to the form (1.13), and in this way to prove the **Theorem (1.2)**. Note, that the volume element $d\mathbf{q}$ in (1.13) is defined in the way analogous to the measure $d\mathbf{H}$ in (4.1).

Proof. We consider in detail the calculation of the two-point correlation function ($n = 2$) defined according to (2.1) as

$$(4.6) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \oint \oint \exp [\mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2)] \\ \times \langle \exp -\text{Tr} [\mathbf{H} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)] \rangle_{GUE} \mathcal{D}(\mathbf{z}_1, \xi_1^*) \mathcal{D}(\mathbf{z}_2, \xi_2^*)$$

The case of general n can be verified precisely along the same route without any essential modifications.

The ensemble averaging in the right-hand side of (4.6) is performed according to (4.2), yielding

$$(4.7) \quad \langle \exp -\text{Tr} [\mathbf{H} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)] \rangle_{GUE} = \exp \left[\frac{1}{2N} \text{Tr} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)^2 \right]$$

Now we notice that

$$(4.8) \quad \text{Tr} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)^2 = (\xi_1^* \mathbf{z}_1)^2 + (\xi_2^* \mathbf{z}_2)^2 + 2(\xi_1^* \mathbf{z}_2)(\xi_2^* \mathbf{z}_1)$$

and introducing 2×2 matrix

$$(4.9) \quad \mathbf{a} = \begin{pmatrix} \xi_1^* \mathbf{z}_1 & \xi_1^* \mathbf{z}_2 \\ \xi_2^* \mathbf{z}_1 & \xi_2^* \mathbf{z}_2 \end{pmatrix}$$

we see that

$$(4.10) \quad \langle \exp -Tr [\mathbf{H} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)] \rangle_{GUE} = \exp \left[\frac{1}{2N} Tr \mathbf{a}^2 \right]$$

Thus

$$(4.11) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \oint \oint e^{\mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2) + \frac{1}{2N} Tr \mathbf{a}^2} \mathcal{D}(\mathbf{z}_1, \xi_1^*) \mathcal{D}(\mathbf{z}_2, \xi_2^*)$$

At the next step we again exploit the Hubbard-Stratonovich transformation (cf. (4.4)) :

$$(4.12) \quad \exp \left[\frac{1}{2N} Tr \mathbf{a}^2 \right] = \int e^{-\frac{N}{2} Tr \mathbf{q}^2 - Tr(\mathbf{q} \mathbf{a})} d\mathbf{q}$$

where \mathbf{q} is 2×2 Hermitian matrix

$$(4.13) \quad \mathbf{q} = \begin{pmatrix} q_{11} & q_{12}^{(R)} + i q_{12}^{(I)} \\ q_{12}^{(R)} - i q_{12}^{(I)} & q_{22} \end{pmatrix}, \quad d\mathbf{q} = \frac{N^2}{2\pi^2} dq_{11} dq_{22} dq_{12}^{(R)} dq_{12}^{(I)}$$

The relation (4.12) follows immediately from (4.2) specified for 2×2 matrices, after an appropriate change of notations. Substituting (4.12) back to (4.11) and changing the order of integrations, we arrive at

$$(4.14) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \int e^{-\frac{N}{2} Tr \mathbf{q}^2} d\mathbf{q} \oint \oint e^{\mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2) - Tr(\mathbf{q} \mathbf{a})} \mathcal{D}(\mathbf{z}_1, \xi_1^*) \mathcal{D}(\mathbf{z}_2, \xi_2^*)$$

Now we note that

$$(4.15) \quad \begin{aligned} & \mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2) - Tr(\mathbf{q} \mathbf{a}) \\ &= (\mu_1 - q_{11})(\xi_1^* \mathbf{z}_1) - q_{12}(\xi_2^* \mathbf{z}_1) - \bar{q}_{12}(\xi_1^* \mathbf{z}_2) + (\mu_2 - q_{22})(\xi_2^* \mathbf{z}_2) \\ &\equiv (\xi_1^*, \xi_2^*) \begin{pmatrix} (\mu_1 - q_{11}) \mathbf{1}_N & -\bar{q}_{12} \mathbf{1}_N \\ -q_{12} \mathbf{1}_N & (\mu_2 - q_{22}) \mathbf{1}_N \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} \end{aligned}$$

which allows us to evaluate the contour integrals in (4.14) with help of the formula (2.1). The result is simply

$$(4.16) \quad \begin{aligned} & \oint \oint e^{\mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2) - Tr(\mathbf{q} \mathbf{a})} \mathcal{D}(\mathbf{z}_1, \xi_1^*) \mathcal{D}(\mathbf{z}_2, \xi_2^*) \\ &= Per \begin{pmatrix} (\mu_1 - q_{11}) \mathbf{1}_N & -\bar{q}_{12} \mathbf{1}_N \\ -q_{12} \mathbf{1}_N & (\mu_2 - q_{22}) \mathbf{1}_N \end{pmatrix} \end{aligned}$$

Using that (i) the permanent of a block-diagonal matrix is equal to the product of the permanents of the diagonal blocks and (ii) the matrix

transposition does not change its permanent we finally arrive at the required identity:

$$(4.17) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \int e^{-\frac{N}{2} Tr \mathbf{q}^2} [Per(\mathbf{M} - \mathbf{q})]^N d\mathbf{q}$$

where $\mathbf{M} = diag(\mu_1, \mu_2)$. \square

According to the Theorem (1.2) the evaluation of the correlation functions of GUE permanental polynomials reduces to evaluation of the integral in the right-hand side of (1.13). In the earlier studied case of determinantal polynomials performing a similar integration was achieved by shifting the integration variables $\mathbf{M} - \mathbf{q} \rightarrow \mathbf{q}$, and passing to the eigenvalue decomposition $\mathbf{q} = \mathbf{U}_n \mathbf{q}_d \mathbf{U}_n^*$, $\mathbf{q}_d = diag(q_1, \dots, q_n)$, $\mathbf{U}_n \in U(n)$. As $\det \mathbf{q} = \det \mathbf{q}_d$ (and is therefore independent of \mathbf{U}_n) one can evaluate the unitary-group integral with the help of HCIZ identity which results in a great simplification. Unfortunately, permanents lack the nice invariance properties of the determinants, and for general $n > 2$ the exact evaluation of the corresponding integrals is still outstanding. The case $n = 2$ turns out however to be specific, and a simple modification of the procedure described above does yield the full solution of the problem.

To this end, we note that for 2×2 Hermitian matrices \mathbf{q} specified in (4.13)

$$\begin{aligned} Per(\mathbf{M} - \mathbf{q}) &= (\mu_1 - q_{11})(\mu_2 - q_{22}) + \bar{q}_{12}q_{12} \\ &= (-1) \det \begin{pmatrix} (-\mu_1 + q_{11}) & -\bar{q}_{12} \\ -q_{12} & (\mu_2 - q_{22}) \end{pmatrix} \end{aligned}$$

Substituting this expression to the right-hand side of (4.17), and changing $q_{11} \rightarrow -q_{11}$ (which obviously leaves $Tr \mathbf{q}^2$ invariant), we therefore see that

$$(4.18) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = (-1)^N \int e^{-\frac{N}{2} Tr \mathbf{q}^2} [\det(\tilde{\mathbf{M}} - \mathbf{q})]^N d\mathbf{q}$$

where $\tilde{\mathbf{M}} = diag(-\mu_1, \mu_2)$. Now, as explained above, we can shift the integration variables $\tilde{\mathbf{M}} - \mathbf{q} \rightarrow \mathbf{q}$, and pass to the eigenvalue decomposition $\mathbf{q} = \mathbf{U}_2 \mathbf{q}_d \mathbf{U}_2^*$, $\mathbf{q}_d = diag(q_1, q_2)$, $\mathbf{U}_2 \in U(2)$. The measure is changed as $d\mathbf{q} \rightarrow (N^2/4\pi)(q_1 - q_2)^2 dq_1 dq_2 d\mu_2(U)$, and

$$Tr \mathbf{q}^2 \rightarrow Tr(\tilde{\mathbf{M}} - \mathbf{q})^2 = (\mu_1^2 + \mu_2^2) - 2Tr(\tilde{\mathbf{M}} \mathbf{U}_2 \mathbf{q}_d \mathbf{U}_2^*) + (q_1^2 + q_2^2).$$

Combining all these facts, we find that

$$(4.19) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \frac{N^2}{4\pi} e^{-\frac{N}{2}(\mu_1^2 + \mu_2^2)} \\ \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{N}{2}(q_1^2 + q_2^2)} (q_1 q_2)^N (q_1 - q_2)^2 dq_1 dq_2 I_{HCIZ}(q_1, q_2)$$

where we defined

$$(4.20) \quad I_{HCIZ}(q_1, q_2) = \int \exp -N \text{Tr} \left(\tilde{\mathbf{M}} \mathbf{U}_2 \mathbf{q}_d \mathbf{U}_2^* \right) d\mu_2(U) \\ = \frac{1}{N(q_1 - q_2)(\mu_1 + \mu_2)} \det \begin{pmatrix} e^{N\mu_1 q_1} & e^{-N\mu_2 q_1} \\ e^{N\mu_1 q_2} & e^{-N\mu_2 q_2} \end{pmatrix}$$

according to (3.5) as the standard HCIZ integral. After straightforward algebraic manipulations the formulae (4.19)-(4.20) give rise to the following final expression for the two-point correlation function:

$$(4.21) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GUE} = \frac{1}{(-i\mu_1) - (i\mu_2)} \det \begin{pmatrix} \pi_N(-i\mu_1) & \pi_N(i\mu_2) \\ \pi_{N+1}(-i\mu_1) & \pi_{N+1}(i\mu_2) \end{pmatrix},$$

where we have used the integral representation (3.21) for the monic Hermite polynomials.

In particular, last expression proves a curious relation (1.15) between correlation functions of permanental and characteristic polynomials of GUE matrices, valid in view of the corresponding Brezin-Hikami result (1.10).

5. TWO-POINT CORRELATION FUNCTION OF THE PERMANENTAL POLYNOMIALS FOR MATRICES FROM THE CIRCULAR UNITARY ENSEMBLE

Consider the space \mathcal{U} (identified with the unitary group $U(N)$) of $N \times N$ unitary matrices, satisfying $\mathbf{U}^{-1} = \mathbf{U}^*$, with the probability measure on \mathcal{U} being chosen according to the Haar's measure on the unitary group. This condition defines the Circular Unitary Ensemble (CUE). With the permanental polynomial defined as $p(\mu) = \text{Per}(\mu \mathbf{1}_N - \mathbf{U})$, it is immediately clear that any non-trivial correlation function must have the form of a product of an even number of permanental polynomials, precisely half of the factors involving matrices \mathbf{U}^* , that is

$$(5.1) \quad \left\langle \prod_{k=1}^n p\left(\mu_k^{(A)}\right) \overline{p\left(\mu_k^{(B)}\right)} \right\rangle_{CUE}$$

In what follows we will address the simplest non-trivial case of the two-point function ($n = 1$), which according to (2.1) has the following

integral representation:

$$(5.2) \quad \left\langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \right\rangle_{CUE} = \oint \oint e^{\left[\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \overline{\mu^{(B)}}(\xi_B^* \mathbf{z}_B) \right]} \times \left\langle e^{-Tr[\mathbf{U}(\mathbf{z}_A \otimes \xi_A^*) + \mathbf{U}^*(\mathbf{z}_B \otimes \xi_B^*)]} \right\rangle_{CUE} \mathcal{D}(\mathbf{z}_A, \xi_A^*) \mathcal{D}(\mathbf{z}_B, \xi_B^*)$$

To perform the ensemble average of the exponential factor in the integrand above we make use of the following

Theorem 5.1. *For two general complex square $N \times N$ matrices \mathbf{A} and \mathbf{B} define*

$$(5.3) \quad F(\mathbf{A}\mathbf{B}^*) = \int_{U(N)} \exp Tr[\mathbf{A}\mathbf{U} + \mathbf{U}^*\mathbf{B}^*] d\mu_N(U)$$

Suppose $\mathbf{A}\mathbf{B}^$ has rank 1. Then for $N \geq 2$ holds*

$$(5.4) \quad F(\mathbf{A}\mathbf{B}^*) = (N-1) \int_0^1 (1-t)^{N-2} I_0(2\sqrt{tv^2}) dt$$

where v^2 is the only non-zero eigenvalue of $\mathbf{A}\mathbf{B}^$ and*

$$(5.5) \quad I_0(z) = \sum_{j=0}^{\infty} \frac{z^{2j}}{2^{2j} j!^2}$$

is the modified Bessel function.

Proof. The theorem is just a special rank-1 case of a more general statement proved for any rank $2m \leq N$ in [27] using the Schur function expansion. It is intimately related to the so-called "bosonic colour-flavour" transformation introduced by Zirnbauer in [55]. For rank $m = N$ the function $F(\mathbf{A}\mathbf{B}^*)$ was earlier calculated in [52], see also [6]. \square

According to (5.2) we are interested in a particular case $\mathbf{A} = -\mathbf{z}_A \otimes \xi_A^*$ and $\mathbf{B}^* = -\mathbf{z}_B \otimes \xi_B^*$, so that $\mathbf{A}\mathbf{B}^* = (\xi_A^* \mathbf{z}_B) \mathbf{z}_A \otimes \xi_B^*$, which is obviously of rank 1, with $v^2 = (\xi_A^* \mathbf{z}_B)(\xi_B^* \mathbf{z}_A)$. Using (5.4) and the expansion (5.5) it is convenient to perform integration over t explicitly, and find:

$$(5.6) \quad \left\langle e^{-Tr[\mathbf{U}(\mathbf{z}_A \otimes \xi_A^*) + \mathbf{U}^*(\mathbf{z}_B \otimes \xi_B^*)]} \right\rangle_{CUE} = (N-1)! \sum_{j=0}^{\infty} \frac{[(\xi_A^* \mathbf{z}_B)(\xi_B^* \mathbf{z}_A)]^j}{j!(N-1+j)!}$$

Substituting this expression to (5.2) we need to evaluate the integral

$$(5.7) \quad I_j(\mu^{(A)}, \overline{\mu^{(B)}}) = \oint \oint e^{\left[\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \overline{\mu^{(B)}}(\xi_B^* \mathbf{z}_B) \right]} [(\xi_A^* \mathbf{z}_B)(\xi_B^* \mathbf{z}_A)]^j \mathcal{D}(\mathbf{z}_A, \xi_A^*) \mathcal{D}(\mathbf{z}_B, \xi_B^*)$$

This is most easily done by first considering the generating function

$$(5.8) \quad I_j(\mu^{(A)}, \overline{\mu^{(B)}}, p_1, p_2) \\ = \oint \oint e^{\left[\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \overline{\mu^{(B)}}(\xi_B^* \mathbf{z}_B) + p_1(\xi_A^* \mathbf{z}_B) + p_2(\xi_B^* \mathbf{z}_A) \right]} \mathcal{D}(\mathbf{z}_A, \xi_A^*) \mathcal{D}(\mathbf{z}_B, \xi_B^*)$$

from which (5.7) is obtained by a simple differentiation, and then taking limit $p_1 = p_2 = 0$. Noticing

$$(5.9) \quad \mu^{(A)}(\xi_A^* \mathbf{z}_A) + \overline{\mu^{(B)}}(\xi_B^* \mathbf{z}_B) + p_1(\xi_A^* \mathbf{z}_B) + p_2(\xi_B^* \mathbf{z}_A) \\ \equiv (\xi_A^*, \xi_B^*) \begin{pmatrix} \mu^{(A)} \mathbf{1}_N & p_1 \mathbf{1}_N \\ p_2 \mathbf{1}_N & \overline{\mu^{(B)}} \mathbf{1}_N \end{pmatrix} \begin{pmatrix} \mathbf{z}_A \\ \mathbf{z}_B \end{pmatrix}$$

we can apply the Lemma (2.1) to (5.8) and find

$$(5.10) \quad I_j(\mu^{(A)}, \overline{\mu^{(B)}}, p_1, p_2) = \left[\text{Per} \begin{pmatrix} \mu^{(A)} & p_1 \\ p_2 & \overline{\mu^{(B)}} \end{pmatrix} \right]^N,$$

where we have used that the permanent of a block-diagonal matrix is equal to the product of the permanents of the diagonal blocks. Now a simple differentiation yields the required result:

$$(5.11) \quad I_j(\mu^{(A)}, \overline{\mu^{(B)}}) = \begin{cases} \frac{N!j!}{(N-j)!} (\mu^{(A)} \overline{\mu^{(B)}})^{N-j}, & j \leq N \\ 0, & j > N \end{cases}$$

Combining all these results, and changing $(N-j) \rightarrow j$ for convenience we arrive at the final expression for the two-point correlation function (5.2):

$$(5.12) \quad \left\langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \right\rangle_{CUE} = N!(N-1)! \sum_{j=0}^N \frac{(\mu^{(A)} \overline{\mu^{(B)}})^j}{j!(2N-1-j)!}$$

A direct check shows that the above expression can also be written in the form of an integral:

$$(5.13) \quad \left\langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \right\rangle_{CUE} = (N-1) \int_0^1 (1-t)^{N-2} (\mu^{(A)} \overline{\mu^{(B)}} + t)^N dt$$

This latter form is especially suitable for extracting the asymptotic behaviour in the limit of large N by the Laplace method. In particular, for $\mu^{(A)} = \mu^{(B)} = \mu$ the integral is dominated by the vicinity of the point of maximum of the integrand $t = t_m = (1 - |\mu|^2)/2$ for $|\mu| < 1$, and by the lower limit of integration $t = 0$ for $|\mu| \geq 1$. A straightforward

calculation then shows that

(5.14)

$$\Phi_p(|\mu|^2) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |p(\mu)|^2 \rangle_{CUE} = \begin{cases} 2 \ln \left(\frac{1+|\mu|^2}{2} \right), & |\mu| < 1 \\ \ln |\mu|^2, & |\mu| \geq 1 \end{cases}$$

Under assumption that the relation (1.18) holds for CUE matrices, the expression (5.14) for Φ_p yields after applying to it the Laplace operator a two-dimensional density profile in the complex plane $\mu = x + iy$ given in the equation (1.21).

Since this result is very different from the density of (unimodular) eigenvalues of the CUE matrices, let us briefly show how to recover the latter density in a similar approach. The analogue of (5.13) was found in [27], and reads

(5.15)

$$\left\langle d(\mu^{(A)}) \overline{d(\mu^{(B)})} \right\rangle_{CUE} = (N+1) \int_0^\infty \frac{1}{(1+t)^{N+2}} (\mu^{(A)} \overline{\mu^{(B)}} + t)^N dt$$

where $d(\mu) = \det(\mu \mathbf{1}_N - \mathbf{U})$ is the corresponding characteristic polynomial. In contrast to the previously considered case, in the limit of large N the integrand for $\mu^{(A)} = \mu^{(B)} = \mu$ does not have a maximum in the domain of integration, and thus the integral is dominated by the vicinity of the upper limit $t = \infty$ for $|\mu| < 1$, and by the vicinity of the lower limit $t = 0$ for $|\mu| > 1$. This yields

$$(5.16) \quad \Phi_d(|\mu|^2) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |d(\mu)|^2 \rangle_{CUE} = \begin{cases} 0, & |\mu| < 1 \\ \ln |\mu|^2, & |\mu| > 1 \end{cases}$$

We see, that the function $\Phi_d(x, y)$ is harmonic everywhere in the complex plane $\mu = x + iy$, except the unit circle $|\mu| = 1$. This means that the density of eigenvalues of CUE matrices is indeed vanishing everywhere except $|z| = 1$, in agreement with obvious unimodularity of the corresponding eigenvalues. The discontinuity of the normal derivative of the "potential" Φ_1 across the unit circle yields the constant density on the circle, as expected.

6. CORRELATION FUNCTIONS FOR PERMANENTAL POLYNOMIALS OF GINIBRE MATRICES

The joint probability density of matrix entries for complex Ginibre $N \times N$ matrices \mathbf{Z} is given by

$$(6.1) \quad \mathcal{P}_{GUE}(\mathbf{Z}) d\mathbf{Z} d\mathbf{Z}^* = C_{Gin} \prod_{i,j}^N e^{-N \bar{Z}_{ij} Z_{ij}} \prod_{i,j}^N dZ_{ij}^R dZ_{ij}^I$$

where $C_{Gin} = (N/\pi)^{N^2}$ is the corresponding normalization constant.

Lemma 6.1. *The identity*

$$(6.2) \quad \langle e^{-\text{Tr}(\mathbf{Z}\mathbf{A}+\mathbf{B}\mathbf{Z}^*)} \rangle_{\text{Gin}} = e^{\frac{1}{N}\text{Tr}\mathbf{A}\mathbf{B}}$$

holds for any two general $N \times N$ matrices \mathbf{A}, \mathbf{B} with complex entries.

Proof. The integral in the left hand side factorizes into a product of simple Gaussian integrals

$$\int e^{-N\bar{Z}_{ij}Z_{ij}-Z_{ij}A_{ji}-\bar{Z}_{ij}B_{ij}} dZ_{ij}^{(R)} dZ_{ij}^{(I)} = \frac{\pi}{N} e^{\frac{1}{N}A_{ji}B_{ij}}$$

The product of all those factors together with the normalization constant C_{GUE} yields the right-hand side of (6.2). □

Again, as in the case of CUE the nontrivial correlation functions for Ginibre ensemble must contain even number of factors, with half of them being permanental polynomials of Z , the rest being permanental polynomials of Z^* . More precisely, we will prove the following

Theorem 6.2. *For any integer $n = 1, 2, \dots$ the $2n$ -point correlation function of permanental polynomials of Ginibre matrices \mathbf{Z} has the following integral representation:*

$$(6.3) \quad \left\langle \prod_{k=1}^n p\left(\mu_k^{(A)}\right) \overline{p\left(\mu_k^{(B)}\right)} \right\rangle_{\text{Gin}} = \int e^{-N\text{Tr}\mathbf{Q}^2} \left[\text{Per} \begin{pmatrix} \mathbf{M}_A & \mathbf{Q}^* \\ \mathbf{Q} & \mathbf{M}_B \end{pmatrix} \right]^N d\mathbf{Q} d\mathbf{Q}^*$$

In this expression $\mathbf{M}_A = \text{diag}(\mu_1^{(A)}, \dots, \mu_n^{(A)})$ and $\mathbf{M}_B = \text{diag}(\bar{\mu}_1^{(B)}, \dots, \bar{\mu}_n^{(B)})$ are $n \times n$ diagonal matrices and \mathbf{Q} is a general $n \times n$ complex matrix, with the volume element $d\mathbf{Q} d\mathbf{Q}^*$ defined in the way analogous to the measure $d\mathbf{Z} d\mathbf{Z}^*$ in (6.1).

Proof. We consider in detail the calculation of the two-point correlation function ($n = 1$) defined according to (2.1) as

$$(6.4) \quad \left\langle p\left(\mu^{(A)}\right) \overline{p\left(\mu^{(B)}\right)} \right\rangle_{\text{Gin}} = \oint \oint \exp \left[\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \bar{\mu}^{(B)}(\xi_B^* \mathbf{z}_B) \right] \\ \times \langle \exp -\text{Tr} [\mathbf{Z} \mathbf{z}_A \otimes \xi_A^* + \mathbf{Z}^* \mathbf{z}_B \otimes \xi_B^*] \rangle_{\text{Gin}} \mathcal{D}(\mathbf{z}_A, \xi_A^*) \mathcal{D}(\mathbf{z}_B, \xi_B^*)$$

The ensemble averaging in the right-hand side of (4.6) is performed according to (6.2), and using $\text{Tr}(\mathbf{z}_A \otimes \xi_A^*)(\mathbf{z}_B \otimes \xi_B^*) = (\xi_A^* \mathbf{z}_B)(\xi_B^* \mathbf{z}_A)$ yielding

$$(6.5) \quad \langle \exp -\text{Tr} [\mathbf{Z} \mathbf{z}_A \otimes \bar{\xi}_A + \mathbf{Z}^* \mathbf{z}_B \otimes \bar{\xi}_B] \rangle_{\text{Gin}} = \exp \left[\frac{1}{N} (\xi_A^* \mathbf{z}_B)(\xi_B^* \mathbf{z}_A) \right]$$

Introducing the notations

$$(6.6) \quad P_{AB} = (\xi_A^* \mathbf{z}_B), \quad P_{BA} = (\xi_B^* \mathbf{z}_A)$$

we notice that the relevant form of the Hubbard-Stratonovich transformation in this case is given by (cf. (6.2))

$$(6.7) \quad e^{\frac{1}{N}[P_{AB}P_{BA}]} = \frac{N}{\pi} \int \int e^{-N\bar{Q}Q + QP_{AB} + \bar{Q}P_{BA}} dQ^{(R)} dQ^{(I)}$$

where the integral over Q is in the plane of a general complex variable, $Q = Q^R + iQ^I$. Substituting this back to (6.5) and then to (6.4) and changing the order of integrations, we have

$$\begin{aligned} \left\langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \right\rangle_{\text{Gin}} &= \frac{N}{\pi} \int \int e^{-N\bar{Q}Q} dQ d\bar{Q} \\ &\times \oint \oint \exp [\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \bar{\mu}^{(B)}(\xi_B^* \mathbf{z}_B) + Q(\xi_A^* \mathbf{z}_B) + \bar{Q}(\xi_B^* \mathbf{z}_A)] \mathcal{D}(\mathbf{z}_B, \xi_B^*) \mathcal{D}(\mathbf{z}_B, \xi_B^*). \end{aligned}$$

Further noticing that

$$\mu^{(A)}(\xi_A^* \mathbf{z}_A) + \bar{\mu}^{(B)}(\xi_B^* \mathbf{z}_B) + Q(\xi_A^* \mathbf{z}_B) + \bar{Q}(\xi_B^* \mathbf{z}_A) = (\xi_A^*, \xi_B^*) \begin{pmatrix} \mu^{(A)} \mathbf{1}_N & \bar{Q} \mathbf{1}_N \\ Q \mathbf{1}_N & \bar{\mu}^{(B)} \mathbf{1}_N \end{pmatrix} \begin{pmatrix} \mathbf{z}_A \\ \mathbf{z}_B \end{pmatrix}$$

the contour integral can be performed using (2.1), yielding finally

$$\begin{aligned} \left\langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \right\rangle_{\text{Gin}} &= \frac{N}{\pi} \int \int e^{-N\bar{Q}Q} \text{Per} \begin{pmatrix} \mu^{(A)} \mathbf{1}_N & \bar{Q} \mathbf{1}_N \\ Q \mathbf{1}_N & \bar{\mu}^{(B)} \mathbf{1}_N \end{pmatrix} dQ d\bar{Q} \\ (6.8) \quad &= \frac{N}{\pi} \int \int e^{-N\bar{Q}Q} \left[\text{Per} \begin{pmatrix} \mu^{(A)} & \bar{Q} \\ Q & \bar{\mu}^{(B)} \end{pmatrix} \right]^N dQ d\bar{Q} \end{aligned}$$

in accordance with (6.3).

The proof for general n follows precisely the same lines with minimal changes. The place of two parameters P_{AB} and P_{BA} defined in (6.6) is taken by two $n \times n$ matrices \mathbf{P}_{AB} and \mathbf{P}_{BA} with entries

$$(6.9) \quad [P_{AB}]_{ij} = (\xi_{A,i}^* \mathbf{z}_{B,j}), \quad [P_{BA}]_{ij} = (\xi_{B,i}^* \mathbf{z}_{A,j})$$

where $i, j = 1, \dots, n$, and the corresponding Hubbard-Stratonovich transformation is

$$(6.10) \quad e^{\frac{1}{N} \text{Tr}[\mathbf{P}_{AB} \mathbf{P}_{BA}]} = \int e^{-N \text{Tr}[\mathbf{Q}^* \mathbf{Q}] + \text{Tr}[\mathbf{Q} \mathbf{P}_{AB} + \mathbf{Q}^* \mathbf{P}_{BA}]} d\mathbf{Q} d\mathbf{Q}^*$$

□

Note: The problem of evaluating the integral in the right-hand side of (6.3) for general $n > 1$ is still outstanding. For $n = 1$ the calculation is elementary:

$$\begin{aligned} \langle p(\mu^{(A)}) \overline{p(\mu^{(B)})} \rangle_{Gin} &= \frac{N}{\pi} \int \int e^{-N\overline{Q}Q} [\mu^{(A)}\overline{\mu}^{(B)} + \overline{Q}Q]^N dQ d\overline{Q} \\ (6.11) \quad &= N \int_0^\infty e^{-NR} (\mu^{(A)}\overline{\mu}^{(B)} + R)^N dR \end{aligned}$$

$$(6.12) \quad = \frac{N!}{N^N} \sum_{k=0}^N \frac{1}{k!} [N\mu^{(A)}\overline{\mu}^{(B)}]^k$$

The integral form (6.11) is most convenient for extracting the large- N asymptotics by the Laplace method. For $\mu^{(A)} = \mu^{(B)} = \mu$ the integrand has a sharp maximum around $R = 1 - |\mu|^2$ as long as $|\mu| < 1$ and is dominated by the lower limit $R = 0$ for $|\mu| > 1$. This gives:

$$(6.13) \quad \Phi_{Gin}(|\mu|^2) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \langle |p(\mu)|^2 \rangle_{Gin} = \begin{cases} |\mu|^2 - 1, & |\mu| < 1 \\ \ln |\mu|^2, & |\mu| \geq 1 \end{cases}$$

Under further assumption of validity of (1.18) such an expression implies the uniform density of permanental roots inside the unit circle, as conjectured in (1.22).

7. CORRELATION FUNCTIONS FOR PERMANENTAL POLYNOMIALS OF GOE MATRICES

The joint probability density of matrix entries for GOE $N \times N$ real symmetric matrices \mathbf{H} is given by

$$(7.1) \quad \mathcal{P}_{GOE}(\mathbf{H}) d\mathbf{H} = C_{GOE} \prod_{i=1}^N e^{-\frac{N}{2} H_{ii}^2} \prod_{i < j} e^{-N H_{ij}^2} \prod_{i=1}^N dH_{ii} \prod_{i < j} dH_{ij}$$

where $C_{GOE} = 2^{-\frac{N}{2}} (N/\pi)^{N(N+1)/4}$ is the corresponding normalization constant.

It is straightforward to verify the following

Lemma 7.1. *The identity*

$$(7.2) \quad \langle e^{-\text{Tr} \mathbf{H} \mathbf{A}} \rangle_{GOE} = e^{\frac{1}{4N} [\text{Tr} \mathbf{A}^2 + \text{Tr} \mathbf{A} \mathbf{A}^T]}$$

holds for any general $N \times N$ matrix \mathbf{A} with complex entries.

The use of this lemma allows one to represent the ensemble averaging of the GOE permanental polynomial $p(\mu)$ from (3.13) in the form (7.3)

$$\langle p(\mu) \rangle_{GOE} = \oint \exp \left[\mu(\xi^* \mathbf{z}) + \frac{1}{4N} \left(\sum_i^N \bar{\xi}_i z_i \right)^2 + \frac{1}{4N} \sum_i^N \bar{\xi}_i^2 \sum_i^N z_i^2 \right] \mathcal{D}(\mathbf{z}, \bar{\xi})$$

where we used that $Tr(\mathbf{z} \otimes \xi^*)(\mathbf{z} \otimes \xi^*)^T = \sum_i \bar{\xi}_i^2 \sum_i z_i^2$.

Following the same strategy as for GUE case, we use the appropriate gaussian integrals ("Hubbard-Stratonovich transformation") to linearize the quartic terms in the exponential. The first term is identical (up to the factor 1/2) to that in the GUE case, and we can use (4.4). The second term in the exponential is dealt with exploiting (6.7), with the correspondence $P_{AB} \rightarrow \sum_i \bar{\xi}_i^2$, $P_{BA} \rightarrow \sum_i z_i^2$. Exchanging the order of integrations, and remembering that $\mathcal{D}(\mathbf{z}, \xi^*)$ is essentially given by $\frac{1}{(2\pi)^{2N}} \prod_{k=1}^N \frac{dz_k}{z_k^2} \prod_{k=1}^N \frac{d\bar{\xi}_k}{\bar{\xi}_k^2}$ (see (2.3)), we arrive at

$$(7.4) \quad \langle p(\mu) \rangle_{GOE} = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{N}{2}q^2} dq \frac{N}{\pi} \int e^{-N\bar{q}_1 q_1} d\bar{q}_1 dq_1 \\ \times \left[\frac{1}{(2\pi)^2} \oint_{|z|=1} \oint_{|\xi|=1} e^{[(\mu - \frac{1}{\sqrt{2}}q)\bar{\xi}z - \frac{1}{2}q_1 z^2 - \frac{1}{2}\bar{q}_1 \bar{\xi}^2]} \frac{dz d\bar{\xi}}{z^2 \bar{\xi}^2} \right]^N$$

Noticing, that the result of the contour integration is independent of q_1, \bar{q}_1 (as any analytic factors depending on z^2 or $\bar{\xi}^2$ could be replaced by zero's term in their Taylor expansion) we see that second line in (7.4) obviously produces $(\mu - \frac{1}{\sqrt{2}}q)^N$, and after simple manipulations and exploitation of (3.21) we arrive at the following

Proposition 7.2. *The expectation value of the permanental polynomial for the Gaussian Orthogonal Ensemble (GOE) is given in terms of the monic Hermite polynomial (3.21) by*

$$(7.5) \quad \langle p(\mu) \rangle_{GOE} = \frac{i^N}{2^{N/2}} \pi_N(-i\sqrt{2}\mu) = i^N \langle d(-i\mu) \rangle_{GOE},$$

where again $d(\mu) = \det(\mu \mathbf{1}_N - H)$. The second equality follows from comparison with the equation (48) of [12].

The same method can be used, *mutatis mutandis*, for evaluating higher correlation function, although actual calculations are much more cumbersome than for the GUE case. Below we restrict ourselves to considering explicitly the two-point correlation function, $n = 2$. As the

starting expression we use the analogue of (4.6):

$$(7.6) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GOE} = \oint \oint \exp [\mu_1(\xi_1^* \mathbf{z}_1) + \mu_2(\xi_2^* \mathbf{z}_2)] \\ \times \langle \exp -Tr [\mathbf{H} (\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)] \rangle_{GOE} \mathcal{D}(\mathbf{z}_1, \xi_1^*) \mathcal{D}(\mathbf{z}_2, \xi_2^*)$$

where the ensemble average is now performed using the identity (7.2), with the role of \mathbf{A} played by the matrix $\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*$.

According to (4.8), for such a matrix $Tr \mathbf{A}^2 = Tr \mathbf{a}^2$, where \mathbf{a} was introduced in (4.9). We also notice that

$$Tr \mathbf{A} \mathbf{A}^T = \sum_{i=1}^N \bar{\xi}_{1i}^2 \sum_{i=1}^N z_{1i}^2 + \sum_{i=1}^N \bar{\xi}_{2i}^2 \sum_{i=1}^N z_{2i}^2 + 2 \sum_{i=1}^N \bar{\xi}_{1i} \bar{\xi}_{2i} \sum_{i=1}^N z_{1i} z_{2i}$$

Combining Hubbard-Stratonovich transformations (4.12) with (three times) (4.4) we see that the ensemble average featuring in (7.6) can be represented as

$$\left\langle e^{-Tr[\mathbf{H}(\mathbf{z}_1 \otimes \xi_1^* + \mathbf{z}_2 \otimes \xi_2^*)]} \right\rangle_{GOE} = \frac{2N^3}{\pi^3} \int e^{-N Tr \mathbf{q}^2 - Tr(\mathbf{q} \mathbf{a})} d\mathbf{q} \int e^{-N(\bar{q}_1 q_1 + \bar{q}_2 q_2 + 2\bar{q}_3 q_3)} \\ \times e^{\frac{1}{2} \sum_{k=1}^2 [q_k \sum_{i=1}^N z_{ki}^2 + \bar{q}_k \sum_{i=1}^N \bar{\xi}_{ki}^2 + q_3 \sum_{i=1}^N z_{1i} z_{2i} + \bar{q}_3 \sum_{i=1}^N \bar{\xi}_{1i} \bar{\xi}_{2i}]} d\bar{q}_1 dq_1 d\bar{q}_2 dq_2 d\bar{q}_3 dq_3$$

Substituting this back to (7.6), changing the order of integration, and suppressing the terms which do not contribute to the value of the contour integral, we obtain after a simple manipulation:

$$(7.7) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GOE} \\ = \frac{2N}{\pi} \int e^{-N(Tr \mathbf{q}^2 + 2\bar{q}_3 q_3)} [\mathcal{J}(\mathbf{q}, \bar{q}_3, q_3; \mu_1, \mu_2)]^N d\mathbf{q} d\bar{q}_3 dq_3$$

where $Tr \mathbf{q}^2 = q_{11}^2 + q_{22}^2 + 2\bar{q}_{12} q_{12}$ and

$$(7.8) \mathcal{J}_N(\mathbf{q}, \bar{q}_3, q_3; \mu_1, \mu_2) = \frac{1}{(2\pi)^4} \oint_{|z_1|=1} \oint_{|\xi_1|=1} \oint_{|z_2|=1} \oint_{|\xi_2|=1} e^{\bar{q}_3 \bar{\xi}_1 \bar{\xi}_2 + q_3 z_1 z_2} \\ \times \exp \left[(\bar{\xi}_1, \bar{\xi}_2) \begin{pmatrix} \mu_1 - q_{11} & -\bar{q}_{12} \\ -q_{12} & \mu_2 - q_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] \prod_{k=1}^2 \frac{dz_k}{z_k^2} \prod_{k=1}^2 \frac{d\bar{\xi}_k}{\bar{\xi}_k^2} \\ = [(\mu_1 - q_{11})(\mu_2 - q_{22}) + \bar{q}_{12} q_{12} + \bar{q}_3 q_3]$$

Changing finally $q_{11} \rightarrow -q_{11}$ we arrive at the final expression

$$(7.9) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GOE} = \frac{2N}{\pi} \int e^{-N(Tr \mathbf{q}^2 + 2\bar{q}_3 q_3)} \\ \times [(\mu_1 + q_{11})(\mu_2 - q_{22}) + \bar{q}_{12} q_{12} + \bar{q}_3 q_3]^N d\mathbf{q} d\bar{q}_3 dq_3$$

Comparing this result with the equation (43) of [12] we infer the validity of the relation:

$$(7.10) \quad \langle p(\mu_1) p(\mu_2) \rangle_{GOE} = \langle d(i\mu_1) d(-i\mu_2) \rangle_{GOE},$$

which is precisely the same as for GUE case, (1.15).

APPENDIX A. PERMANENTS AS MULTIVARIATE GAUSSIAN INTEGRALS

We start with verifying the following

Lemma A.1. *Let $\mathbf{F} > 0$ be positive definite $N \times N$ Hermitian matrix, $\mathbf{a} = (a_1, \dots, a_N)^T$, $\mathbf{b} = (b_1, \dots, b_N)^T$, $\mathbf{s} = (s_1, \dots, s_N)^T$ all be N -component complex vectors, and $d\mathbf{s}^* d\mathbf{s} = \prod_{i=1}^N d\text{Re}(s_i) d\text{Im}(s_i)$. Then the following identity holds*

$$(A.1) \quad e^{\mathbf{a}^* \mathbf{F}^{-1} \mathbf{b}} = \det \mathbf{F} \int e^{-\mathbf{s}^* \mathbf{F} \mathbf{s} - (\mathbf{a}^* \mathbf{s}) - (\mathbf{s}^* \mathbf{b})} d\mathbf{s}^* d\mathbf{s}$$

Proof. Representing $\mathbf{F} = \mathbf{U} \mathbf{f} \mathbf{U}^*$ in terms of the diagonal matrix of the eigenvalues $\mathbf{f} = \text{diag}(f_1, \dots, f_N) > 0$ and the unitary matrix \mathbf{U} of the eigenvectors, and introducing new variables of integration $\mathbf{q} = \mathbf{U}^* \mathbf{s}$ so that $d\mathbf{s}^* d\mathbf{s} = d\mathbf{q}^* d\mathbf{q}$, the N -fold gaussian integral in the right-hand side decomposes into the product of N simple Gaussian integrals of the type (6.7) :

$$(A.2) \quad \prod_{i=1}^N \int e^{-f_i \bar{q}_i q_i - \bar{a}_i q_i - \bar{q}_i b_i} d\bar{q}_i dq_i = \prod_{i=1}^N e^{\frac{1}{f_i} \bar{a}_i b_i}$$

where we introduced the notations $\tilde{\mathbf{a}} = \mathbf{U}^* \mathbf{a}$ and $\tilde{\mathbf{b}} = \mathbf{U}^* \mathbf{b}$. We further find that

$$(A.3) \quad \sum_{i=1}^N \frac{1}{f_i} \bar{a}_i b_i = \sum_{jk} \bar{a}_j \left(\sum_{i=1}^N \mathbf{U}_{ji} \frac{1}{f_i} \mathbf{U}_{ik}^* \right) b_k = \mathbf{a}^* \mathbf{F}^{-1} \mathbf{b}$$

which proves the statement. \square

Proposition A.2. *The permanent of a positive definite Hermitian matrix \mathbf{F} can be represented in terms of a multivariate Gaussian integral as*

$$(A.4) \quad \text{Per } \mathbf{F} = \frac{1}{\det \mathbf{F}} \int e^{-\mathbf{s}^* \mathbf{F}^{-1} \mathbf{s}} \prod_{i=1}^N (\bar{s}_i s_i) d\mathbf{s}^* d\mathbf{s}$$

Proof. According to the contour representation of the permanent (2.1)

$$(A.5) \quad \text{Per } \mathbf{F} = \oint e^{\xi^* \mathbf{F} \mathbf{z}} \mathcal{D}_N(\mathbf{z}, \xi^*).$$

We now replace the exponential factor in the integrand with its representation according to the Lemma (A.1):

$$(A.6) \quad e^{\xi^* \mathbf{F} \mathbf{z}} = \det \mathbf{F}^{-1} \int e^{-\mathbf{s}^* \mathbf{F}^{-1} \mathbf{s} - (\xi^* \mathbf{s}) - (\mathbf{s}^* \mathbf{z})} d\mathbf{s}^* d\mathbf{s}$$

Substituting this back to (A.5), changing the order of integrations and straightforwardly performing the contour integration according to

$$(A.7) \quad \oint e^{-(\xi^* \mathbf{s}) - (\mathbf{s}^* \mathbf{z})} \mathcal{D}_N(\mathbf{z}, \xi^*) = \prod_{i=1}^N (\bar{s}_i s_i)$$

we immediately arrive to (A.4). \square

Note: The formula (A.4) is in fact a variant of the Wick theorem for the Gaussian integrals.

Further note that any positive definite Hermitian matrix \mathbf{F} can be represented as $\mathbf{F} = \mathbf{E} \mathbf{E}^*$, where $\mathbf{E} = \mathbf{U} \mathbf{f}^{1/2}$ in terms of the eigenvectors and the corresponding eigenvalues of \mathbf{F} . This allows us to arrive at the following result, which seems first to appear in [54]:

Corollary A.3. *The permanent of a positive definite Hermitian matrix \mathbf{F} can be represented as*

$$(A.8) \quad \text{Per } \mathbf{F} = \int e^{-\mathbf{v}^* \mathbf{v}} \prod_{i=1}^N ([\mathbf{E} \mathbf{v}^*]_i [\mathbf{E} \mathbf{v}]_i) d\mathbf{v}^* d\mathbf{v}$$

Proof. Introduce in (A.4) a new integration variable \mathbf{v} according to $\mathbf{s} = \mathbf{E} \mathbf{v}$. The integration volume is changed as $d\mathbf{s}^* d\mathbf{s} = \det \mathbf{E} \det \mathbf{E}^* d\mathbf{v}^* d\mathbf{v} \equiv \det \mathbf{F} d\mathbf{v}^* d\mathbf{v}$, and the statement follows. \square

REFERENCES

- [1] G. Akemann, A. Pottier, Ratios of characteristic polynomials in complex matrix models, J.Phys. A37 (2004) L453-L460; math-ph/0404068
- [2] G. Akemann, Y. V.Fyodorov, Universal random matrix correlations of ratios of characteristic polynomials at the spectral edges, Nuclear Phys. B[PM] **664** (2003) 457-476
- [3] A. V. Andreev, B. D. Simons., Correlators of spectral determinants in Quantum Chaos. *Phys. Rev. Lett.* **75** (12), (1995) 2304-2307.
- [4] G. Akemann, G.Vernizzi, Characteristic polynomials of complex random matrix models, Nuclear Phys. B[FS] **660** (2003) 532-556

- [5] J. Baik, P. Deift, and E. Strahov, Products and ratios of characteristic polynomials of random hermitian matrices. *J. Math. Phys.* **44**, (2003) 3657-3670.
- [6] A.B.Balantekin , Character Expansions, Itzykson-Zuber integrals, and QCD partition function, *Phys.Rev.D* **62**, 085017
- [7] F.A. Berezin, Some remarks on the Wigner distribution (in Russian), *Teor.Mat.Fiz.* **17**, (1973), 341-379
- [8] D. Bump and A. Gamburd, On the averages of characteristic polynomials from classical groups, *math-ph/0502043*
- [9] A. Borodin, G. Olshanski, E. Strahov, Giambelli compatible point processes, e-preprint *ArXiv:math-ph/0505021*
- [10] A. Borodin and E. Strahov Averages of characteristic polynomials in Random Matrix Theory, *Commun. Pure and Applied Math.*, **59** (2) , (2006) 161-253
- [11] E. Brezin, S. Hikami, Characteristic polynomials of random matrices, *Commun. Math. Phys.* **214** (2000), 111–135.
- [12] E. Brezin, S. Hikami, Characteristic polynomials of real symmetric random matrices, *Commun. Math. Phys.* **223** (2001) 363–382.
- [13] G. G. Cash , The Permanent Polynomial, *J.Chem.Inf.Comput.Sci.* **40** (2000), 1203-1206
- [14] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, N. C. Snaith, Autocorrelation of random matrix polynomials, *Commun. Math. Phys.* **237** (2003), 365–395.
- [15] J. B. Conrey, P. J. Forrester, and N. C. Snaith, Averages of ratios of characteristic polynomials for the compact classical groups, *IMRN* **7** (2005) 397–431
- [16] J.B. Conrey, D.W. Farmer, M.R. Zirnbauer, Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $U(N)$, e-preprint *ArXiv:math-ph/0511024*
- [17] W. C. Chu , Determinant, permanent, and MacMahon’s Master Theorem, *Lin.Alg. Appl.* **255** (1997), 171-183
- [18] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. Courant lecture notes, 3. New York: Courant Institute of Mathematical sciences, New York University 2000.
- [19] P. Diaconis and A. Gamburd, Random matrices, magic squares and matching polynomials, *Electronic Journal of Combinatorics*, **11**, 2004, R2
- [20] P. Diaconis, R. Graham and S. P. Holmes, Statistical problems involving permutations with restricted positions, (1999), available electronically at <http://www-stat.stanford.edu/~cgates/PERSI/papers/perm8.ps>
- [21] S. Friedland, B. Rider and O. Zeitouni, Concentration of permanent estimators for certain large matrices, *Ann. Appl. Prob.* **14** (2004), 1559–1576
- [22] P. J. Forrester, Log-gases and Random Matrices, (book-in-progress, available from <http://www.ms.unimelb.edu.au/~matpjf/matpjf.html>)
- [23] P. J. Forrester, A. Gamburd, Counting formulas associated with some random matrix averages e-preprint *arXiv:math.CO/0503002*
- [24] Y. V. Fyodorov , Negative moments of characteristic polynomials of random matrices: Ingham-Siegel integral as an alternative to Hubbard-Stratonovich transformation. *Nuclear Phys. B* **621** (2002), No.3, 643-674
- [25] Y. V. Fyodorov, Introduction to the random matrix theory: Gaussian Unitary Ensemble and beyond, (in: *Recent Perspectives in Random Matrix Theory and*

- Number Theory, LMS Lecture Note Series **322**, Cambridge University Press, 2005), pp.31–78
- [26] Y. V. Fyodorov and G. Akemann, On the supersymmetric partition function in QCD-inspired random matrix models, JETP Lett. **77** (2003), 438–441
 - [27] Y. V. Fyodorov and B. A. Khoruzhenko On absolute moments of characteristic polynomials of a certain class of complex random matrices , e-preprint arXiv:math-ph/0602032
 - [28] Y. V. Fyodorov and E. Strahov, An exact formula for general spectral correlation function of random Hermitian matrices. Journ. Phys. A:Math.Gen. **36** (2003) 3203–3213.
 - [29] Y. V. Fyodorov and E. Strahov, Characteristic polynomials of random Hermitian matrices and Duistermaat-Heckman localisation on non-compact Kähler Manifolds, Nucl.Phys.B **630** (2002), 453–491
 - [30] Harish-Chandra , Proc.Nat.Acad.Sci. **42**, (1956) 252
 - [31] C. P. Hughes, J. P. Keating, and N. O’Connell, On the characteristic polynomial of a random unitary matrix. Commun. Math. Phys. **220** (2001) 429–451.
 - [32] C.Itzykson and J.B.Zuber, "The planar approximation. II", J.Math.Phys. **21** (1980), 411
 - [33] J. P. Keating, N. C. Snaith, Random Matrix Theory and $\zeta(1/2+it)$. Commun. Math. Phys. **214** (2000), 57–89.
 - [34] J. P. Keating, N. C. Snaith, Random matrices and L-functions. J. Phys. A. **36** (2003), no. 12, 2859–2881.
 - [35] J. P. Keating, N. C. Snaith, Random matrix theory and L-functions at $s=1/2$, Commun. Math. Phys. **214** (2000), 91–110.
 - [36] O. D. Kellog , Foundations of Potential Theory (Dover Publication, 1954).
 - [37] V. S.Korolyuk, Yu.V.Borovskikh , Random Permanents ("Sluchajnye permanenty", in Russian, Kiev Institute of Mathematics, 1993)
 - [38] A. R. Kräuter , Permanents - a short overview (in German) (Permanenten - ein kurzer Überblick), Seminaire Lotharingien de Combinatoire, **B09b** (1983), 34pp; available electronically at <http://www.emis.ams.org/journals/SLC/opapers/s09kraeu.pdf>
 - [39] A. R.Kräuter , Notes on the solutions of the equation $\text{per}(zI - A) = 0$ (in German), (Einiger Bemerkungen über die Lösungen der Gleichung $\text{per}(zI - A) = 0$), Seminaire Lotharingien de Combinatoire, **B16b** (1987), 10pp; available electronically at <http://www.emis.ams.org/journals/SLC/opapers/s16kraeu.pdf>
 - [40] I., G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford University Press, 1995.
 - [41] M. L. Mehta, Random Matrices, 2nd ed., Academic Press, San Diego, 1991.
 - [42] H. Minc, Permanents, (Addison-Wisley, Reading, Mass. 1978)
 - [43] R. Merris, K. R. Rebman, and W. Watkins, Permanental Polynomials of Graphs, Linear Algebra and App. **38** (1981), 273–278
 - [44] A.Yu.Orlov , Tau Functions and Matrix Integrals, e-preprint arXiv:math-ph/0210012
 - [45] G. A. Rempala and J. Wesolowski, Limit Theorems for random permanents with exchangeable structure, J. Multivar. Anal. **91** (2004), 224–239

- [46] G. A. Rempala and J. Wesolowski, Approximation theorems for random permanents and associated stochastic processes, *Prob. Theor. Rel. Fields* **131** (2004), 442–458
- [47] V. N. Sachkov, *Combinatorial methods in discrete mathematics* (Cambridge University Press, 1996)
- [48] G. Szegő, *Orthogonal Polynomials* (Colloquium Publications, **23** Providence, RI: Am.Math.Soc.) 1975
- [49] E. Strahov. Moments of characteristic polynomials enumerate two-rowed lexicographic arrays, *Electronic Journal of Combinatorics*, **10**, 2003, R24.
- [50] E. Strahov and Y. V. Fyodorov, Universal Results for Correlations of Characteristic Polynomials: Riemann-Hilbert Approach. *Commun. Math. Phys.* **241** (2003) 343–382.
- [51] K. Splittorff K, J.J.M. Verbaarschot, Supersymmetric quenching of the Toda lattice equation, *Nucl. Phys. B* **695** (2004) 84–102
- [52] B. Schlittgen, T. Wettig, Generalization of some integrals over the unitary group, *Journ.Phys.A:Math.Gen.* **36** (2003), 3195–3202
- [53] L. G. Valiant, The complexity of computing the permanent, *Theoret.Comput.Sci.* **8** (1979), 189–201
- [54] J. Wosiek, A simple formula for Bose-Einstein corrections, *Phys.Lett. B* **399** (1997), 130–135
- [55] M. R. Zirnbauer, Supersymmetry for Systems with Unitary Disorder: Circular Ensembles, *Journ.Phys.A:Math.Gen.* **29** (1996), 7113–7136

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM NG7 2RD, UNITED KINGDOM

E-mail address: `yan.fyodorov@nottingham.ac.uk`